Derivatives of the Unit Vector

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Abstract

The first and second derivatives of a time varying normalized (unit) vector are derived. In addition, the partial derivatives of a unit vector with respect to its unnormalized form are presented. The primary purpose is to illustrate how recasting a problem into a different form, even if initially more complicated, can simplify a derivation. The secondary purpose is to demonstrate thinking in terms of projections and other transformations when developing and interpreting algorithms.

1 First Derivative: $\dot{\hat{\mathbf{r}}}$

Given a *position* vector that varies over time, $\mathbf{r}(t)$, and its first derivative, compute the time derivative of the corresponding normalized (*unit*) vector:

$$\frac{d}{dt}\left(\frac{\mathbf{r}\,(t)}{||\mathbf{r}\,(t)||}\right)$$

To ease notational burden, $\mathbf{r}(t)$ and its magnitude $||\mathbf{r}(t)||$ will be written $\mathbf{r}$ and r, respectively. The solution can be approached in the form

$$\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) = \frac{d}{dt}\left(\frac{\mathbf{r}}{\sqrt{r_x^2 + r_y^2 + r_z^2}}\right)$$

where the components of $\mathbf{r}$ are within the radical. Performing the derivation starting with this structure is a good exercise, but tedious and error prone. The lazy (and not dimensionally constrained) method is to re-frame the problem.

$$\begin{aligned}\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) &= \frac{d}{dt}\left(\frac{\mathbf{r}}{\sqrt{\mathbf{r}^{\mathrm{T}}\mathbf{r}}}\right) \\ &= \frac{d}{dt}\left(\mathbf{r}\left(\mathbf{r}^{\mathrm{T}}\mathbf{r}\right)^{-\frac{1}{2}}\right)\end{aligned}$$

The product and chain rules can now be applied to the vector itself without thought to individual components. Using the *dot* notation to indicate the *velocity* of $\mathbf{r}$,

$$\dot{\mathbf{r}} = \frac{d}{dt}\mathbf{r}$$

the derivation can be carried out with less clutter:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) &= \frac{d}{dt}\left(\mathbf{r}\left(\mathbf{r}^{\mathrm{T}}\mathbf{r}\right)^{-\frac{1}{2}}\right) \\ &= \dot{\mathbf{r}}\left(\mathbf{r}^{\mathrm{T}}\mathbf{r}\right)^{-\frac{1}{2}} + \mathbf{r}\frac{d}{dt}\left(\mathbf{r}^{\mathrm{T}}\mathbf{r}\right)^{-\frac{1}{2}} \\ &= \frac{\dot{\mathbf{r}}}{r} + \mathbf{r}\left(-\frac{1}{2}\right)\left(\mathbf{r}^{\mathrm{T}}\mathbf{r}\right)^{-3/2}\frac{d}{dt}\left(\mathbf{r}^{\mathrm{T}}\mathbf{r}\right) \\ &= \frac{\dot{\mathbf{r}}}{r} - \frac{1}{2}\frac{\mathbf{r}}{r^3}\left(\dot{\mathbf{r}}^{\mathrm{T}}\mathbf{r} + \mathbf{r}^{\mathrm{T}}\dot{\mathbf{r}}\right) \\ &= \frac{\dot{\mathbf{r}}}{r} - \frac{\mathbf{r}\mathbf{r}^{\mathrm{T}}}{r^3}\dot{\mathbf{r}} \qquad (1)\end{aligned}$$

This can be reformulated into more basic concepts with the use of the identity matrix I and factoring out $\dot{\mathbf{r}}$.

$$\begin{aligned}\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) &= \frac{\dot{\mathbf{r}}}{r} - \frac{\mathbf{r}\mathbf{r}^{\mathrm{T}}}{r^3}\dot{\mathbf{r}} \\ &= \left(\mathrm{I} - \frac{\mathbf{r}\mathbf{r}^{\mathrm{T}}}{r^2}\right)\frac{\dot{\mathbf{r}}}{r}\end{aligned}$$

Denoting $\hat{\mathbf{r}}$ as the unit vector of $\mathbf{r}$, the final form is:

$$\dot{\hat{\mathbf{r}}} = \frac{1}{r}\left(\mathrm{I} - \hat{\mathbf{r}}\hat{\mathbf{r}}^{\mathrm{T}}\right)\dot{\mathbf{r}} \qquad (2)$$

The rate of change of $\hat{\mathbf{r}}$ is $\dot{\mathbf{r}}$ projected onto the plane normal to $\mathbf{r}$, scaled by the inverse of r. This means $\dot{\hat{\mathbf{r}}}$ is normal to $\hat{\mathbf{r}}$.

$$\dot{\hat{\mathbf{r}}} \perp \hat{\mathbf{r}}$$

2 Second Derivative: $\ddot{\hat{\mathbf{r}}}$

The second derivative of $\hat{\mathbf{r}}$ can be carried out in a similar manner as the first. Begin with recognizing the time derivatives of components composing Eqn. (1) through the same process used in §1:

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{r}\right) &= -\frac{\mathbf{r}^{\mathrm{T}}\dot{\mathbf{r}}}{r^3} \qquad (3) \\ \frac{d}{dt}\left(\frac{1}{r^3}\right) &= -3\frac{\mathbf{r}^{\mathrm{T}}\dot{\mathbf{r}}}{r^5} \qquad (4) \\ \frac{d}{dt}\left(\mathbf{r}\mathbf{r}^{\mathrm{T}}\right) &= \mathbf{r}\dot{\mathbf{r}}^{\mathrm{T}} + \dot{\mathbf{r}}\mathbf{r}^{\mathrm{T}} \qquad (5)\end{aligned}$$

The derivative of $\dot{\hat{\mathbf{r}}}$ can more easily be visualized by putting Eqn. (1) in the form

$$\ddot{\hat{\mathbf{r}}} = \frac{d}{dt}\left(\frac{1}{r}\dot{\mathbf{r}} - \frac{1}{r^3}\left(\mathbf{r}\mathbf{r}^{\mathrm{T}}\right)\dot{\mathbf{r}}\right)$$

Employing the product rule using Eqns. (3) through (5):

$$\ddot{\hat{\mathbf{r}}} = -\frac{\mathbf{r}^{\mathrm{T}}\dot{\mathbf{r}}}{r^3}\dot{\mathbf{r}} + \frac{1}{r}\ddot{\mathbf{r}} + 3\frac{\mathbf{r}^{\mathrm{T}}\dot{\mathbf{r}}}{r^5}\left(\mathbf{r}\mathbf{r}^{\mathrm{T}}\right)\dot{\mathbf{r}} - \frac{1}{r^3}\left(\mathbf{r}\dot{\mathbf{r}}^{\mathrm{T}} + \dot{\mathbf{r}}\mathbf{r}^{\mathrm{T}}\right)\dot{\mathbf{r}} - \frac{1}{r^3}\left(\mathbf{r}\mathbf{r}^{\mathrm{T}}\right)\ddot{\mathbf{r}}$$

Factoring r in the denominator, absorbing one r per $\mathbf{r}$, collecting about the *acceleration* $\ddot{\mathbf{r}}$,

$$\ddot{\hat{\mathbf{r}}} = \frac{1}{r}\left\{\left(\mathrm{I} - \hat{\mathbf{r}}\hat{\mathbf{r}}^{\mathrm{T}}\right)\ddot{\mathbf{r}} + \frac{1}{r}\left[\hat{\mathbf{r}}^{\mathrm{T}}\dot{\mathbf{r}}\left(3\hat{\mathbf{r}}\hat{\mathbf{r}}^{\mathrm{T}} - \mathrm{I}\right)\dot{\mathbf{r}} - \left(\hat{\mathbf{r}}\dot{\mathbf{r}}^{\mathrm{T}} + \dot{\mathbf{r}}\hat{\mathbf{r}}^{\mathrm{T}}\right)\dot{\mathbf{r}}\right]\right\}$$

and taking note that $\dot{r}^2 = \dot{\mathbf{r}}^{\mathrm{T}}\dot{\mathbf{r}}$

$$\begin{aligned}
\ddot{\hat{\mathbf{r}}} &= \frac{1}{r}\left\{\left(\mathrm{I} - \hat{\mathbf{r}}\hat{\mathbf{r}}^{\mathrm{T}}\right)\ddot{\mathbf{r}} + \frac{1}{r}\left[\hat{\mathbf{r}}^{\mathrm{T}}\dot{\mathbf{r}}\left(3\hat{\mathbf{r}}\hat{\mathbf{r}}^{\mathrm{T}} - \mathrm{I}\right)\dot{\mathbf{r}} - \dot{r}^2\hat{\mathbf{r}} - \left(\hat{\mathbf{r}}^{\mathrm{T}}\dot{\mathbf{r}}\right)\dot{\mathbf{r}}\right]\right\} \\
&= \frac{1}{r}\left\{\left(\mathrm{I} - \hat{\mathbf{r}}\hat{\mathbf{r}}^{\mathrm{T}}\right)\ddot{\mathbf{r}} - \frac{1}{r}\left[2\left(\dot{\mathbf{r}}\cdot\hat{\mathbf{r}}\right)\left(\mathrm{I} - \frac{3}{2}\hat{\mathbf{r}}\hat{\mathbf{r}}^{\mathrm{T}}\right)\dot{\mathbf{r}} + \dot{r}^2\hat{\mathbf{r}}\right]\right\} \qquad (6)
\end{aligned}$$

Inspection of Eqn. (6) suggests an order of operation for numerical stability. The acceleration is projected onto the plane normal to the position vector. Therefore, centripetal acceleration does not contribute to $\ddot{\hat{\mathbf{r}}}$ because it acts along $\mathbf{r}$. The contribution of the middle term is dependent on the projection of the velocity vector onto the position vector $(\dot{\mathbf{r}}\cdot\hat{\mathbf{r}})$—zero when the velocity is orthogonal to the position vector. For circular motion, the contribution of these two components is zero. Therefore, combining the two potentially small terms first, as is done in Eqn. (7), is a more numerically sound approach. For the last term, the magnitude of the velocity (squared) is applied against the direction of the position vector and will always be present in a non-static system. It should be included after combining the first two terms.

$$\ddot{\hat{\mathbf{r}}} = \frac{1}{r}\left[\left(\mathrm{I} - \hat{\mathbf{r}}\hat{\mathbf{r}}^{\mathrm{T}}\right)\ddot{\mathbf{r}} - 2\frac{\dot{\mathbf{r}}\cdot\hat{\mathbf{r}}}{r}\left(\mathrm{I} - \frac{3}{2}\hat{\mathbf{r}}\hat{\mathbf{r}}^{\mathrm{T}}\right)\dot{\mathbf{r}} - \frac{\dot{r}^2}{r}\hat{\mathbf{r}}\right] \qquad (7)$$

Note, the first and last components of Eqn. (7) are orthogonal. The direction of the second term is less obvious. The velocity vector $\dot{\mathbf{r}}$ is transformed to be between the reflection w.r.t. $\mathbf{r}$ and the normal to $\mathbf{r}$. The $(\mathrm{I} - 1.5\hat{\mathbf{r}}\hat{\mathbf{r}}^{\mathrm{T}})$ operation is neither a projection nor a reflection (see illustration).

As a point of clarification, position, velocity, and acceleration have been limited to describe $\mathbf{r}$, $\dot{\mathbf{r}}$, and $\ddot{\mathbf{r}}$, respectively. The units are as expected—distance, distance per time, and distance per time^2. In contrast, for $\hat{\mathbf{r}}$, $\dot{\hat{\mathbf{r}}}$, and $\ddot{\hat{\mathbf{r}}}$, this terminology has been avoided as distance units have been normalized out of these vectors.

3 Unit Vector Partials: $\frac{\partial\hat{\mathbf{r}}}{\partial\mathbf{r}}$

The partial derivatives of a unit vector w.r.t. the original vector can also be derived using this technique.

$$\frac{\partial\hat{\mathbf{r}}}{\partial\mathbf{r}} = \frac{\partial}{\partial\mathbf{r}}\left(\mathbf{r}\left(\mathbf{r}^{\mathrm{T}}\mathbf{r}\right)^{-\frac{1}{2}}\right)$$

Careful application of the chain and product rules reduces confusion that may occur when carrying out the partials[1].

$$\begin{aligned}
\frac{\partial \hat{\mathbf{r}}}{\partial \mathbf{r}} &= \left(\frac{\partial}{\partial \mathbf{r}}\mathbf{r}\right)\left(\mathbf{r}^{\mathrm{T}}\mathbf{r}\right)^{-\frac{1}{2}} + \mathbf{r}\frac{\partial}{\partial \mathbf{r}}\left(\mathbf{r}^{\mathrm{T}}\mathbf{r}\right)^{-\frac{1}{2}} \\
&= \frac{1}{r}\mathrm{I} - \frac{\mathbf{r}}{2r^3}\frac{\partial}{\partial \mathbf{r}}\left(\mathbf{r}^{\mathrm{T}}\mathbf{r}\right) \\
&= \frac{1}{r}\mathrm{I} - \frac{\mathbf{r}}{2r^3}2\mathbf{r}^{\mathrm{T}} \\
&= \frac{1}{r}\left(\mathrm{I} - \frac{1}{r^2}\mathbf{r}\mathbf{r}^{\mathrm{T}}\right)
\end{aligned}$$

Finally, another scaled projection emerges[2]:

$$\frac{\partial \hat{\mathbf{r}}}{\partial \mathbf{r}} = \frac{1}{r}\left(\mathrm{I} - \hat{\mathbf{r}}\hat{\mathbf{r}}^{\mathrm{T}}\right)$$

Contemplate the implication of this form—the projection onto the plane normal to $\mathbf{r}$ annihilates a degree of freedom. For example, consider a three-dimensional position vector $\mathbf{r}$ and its associated [3×3] covariance $\Sigma_{\mathbf{r}}$. Mapping $\Sigma_{\mathbf{r}}$ to the normalized form is accomplished via the linear transformation

$$\Sigma_{\hat{\mathbf{r}}} = \left[\frac{\partial \hat{\mathbf{r}}}{\partial \mathbf{r}}\right]\Sigma_{\mathbf{r}}\left[\frac{\partial \hat{\mathbf{r}}}{\partial \mathbf{r}}\right]^{\mathrm{T}}$$

The projection results in a rank deficient covariance. A unit vector is constrained to two degrees of freedom given a third component is derived from any two components.

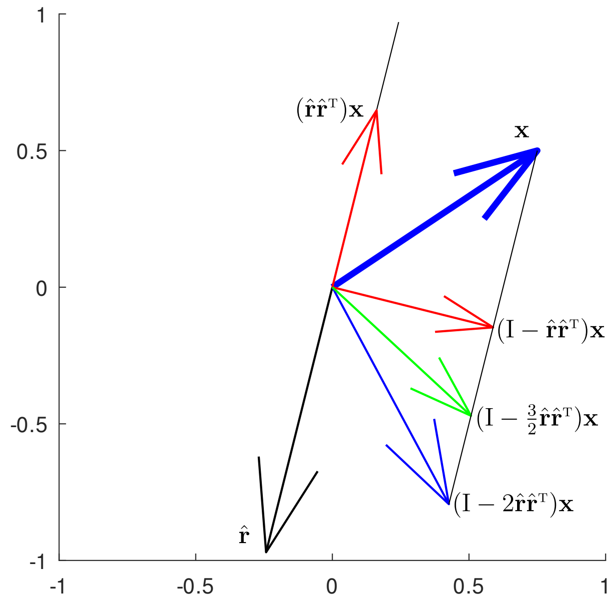
$$\hat{r}_z = \sqrt{1 - \hat{r}_x^2 - \hat{r}_y^2}$$

Therefore, to evaluate the uncertainty in $\hat{\mathbf{r}}$, it is necessary to change the frame of reference through an orthonormal transformation such that $\mathbf{r}$ is aligned with one axis of the new coordinate system. The axes orthogonal to $\mathbf{r}$ contain the surviving information within a [2×2] covariance matrix.

[1] For the derivative of $\hat{\mathbf{r}}^{\mathrm{T}}\hat{\mathbf{r}}$, a component-wise approach may be more clear (while still easily extensible to higher dimensions):

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{r}}\left(\mathbf{r}^{\mathrm{T}}\mathbf{r}\right) &= \frac{\partial}{\partial \mathbf{r}}r^2 \\
&= \frac{\partial}{\partial \mathbf{r}}\left(r_x^2 + r_y^2 + r_z^2\right) \\
&= \begin{bmatrix} 2r_x & 2r_y & 2r_z \end{bmatrix} \\
&= 2\mathbf{r}^{\mathrm{T}}
\end{aligned}$$

[2] Revisit Eqn. (2): $\dot{\hat{\mathbf{r}}} = \frac{\partial \hat{\mathbf{r}}}{\partial \mathbf{r}}\dot{\mathbf{r}}$



Transformations of $\mathbf{x}$ w.r.t. $\hat{\mathbf{r}}$ are illustrated: The projection of $\mathbf{x}$ onto $\hat{\mathbf{r}}$ is $(\hat{\mathbf{r}}\hat{\mathbf{r}}^{\mathrm{T}})\mathbf{x}$ whereas projecting $\mathbf{x}$ onto the plane normal to $\hat{\mathbf{r}}$ is $(\mathrm{I}-\hat{\mathbf{r}}\hat{\mathbf{r}}^{\mathrm{T}})\mathbf{x}$. Reflecting $\mathbf{x}$ w.r.t. $\hat{\mathbf{r}}$ is $(\mathrm{I}-2\hat{\mathbf{r}}\hat{\mathbf{r}}^{\mathrm{T}})\mathbf{x}$. A coefficient from zero to two before $\hat{\mathbf{r}}\hat{\mathbf{r}}^{\mathrm{T}}$ results in $\mathbf{x}$ landing between itself and its reflection. To understand why, note a coefficient of zero is simply $\mathrm{I}\mathbf{x}=\mathbf{x}$. A coefficient of two essentially moves the transformed vector twice the distance (parallel to $\hat{\mathbf{r}}$) as needed for the orthogonal projections. A review of the Householder transformation derivation clarifies the form of both the reflection and orthogonal projections w.r.t. $\mathbf{x}$. For all cases, the vectors resulting from the difference between $\mathbf{x}$ and its transformed states are parallel to $\hat{\mathbf{r}}$.